

CUBICAL n -CATEGORIES AND FINITE-LIMITS THEORIES

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ABSTRACT. This note informally describes a way to build certain cubical n -categories by iterating a process of taking models of certain finite limits theories. We base this discussion on a construction of “double bicategories” as bicategories internal to **Bicat**, and see how to extend this to n -tuple bicategories (and similarly for tricategories etc.) We briefly consider how to reproduce “simpler” definitions of weak cubical n -category from these.

1. INTRODUCTION

This note aims to describe, through a particular example, some relationships between two of the main topics of MakkaiFest: model theory and higher categories. In particular, we aim to describe a way to build certain higher categories by iterating a process of taking models of a theory. The higher categories we have in mind are a particular sort of weak cubical n -category.

While we make no attempt to be comprehensive, we will start with a particular example, an extension of the view of double categories as categories internal to **Cat**. Double categories are structures which have two different category structures on the same set of objects, one “horizontal” and one “vertical”. Often these directions are different, such as the double category whose objects are sets, whose horizontal morphisms are functions, and whose vertical morphisms are relations. There are also “squares” which act as non-identity cells which can fill square diagrams:

$$(1) \quad \begin{array}{ccc} X & \xrightarrow{g} & Y \\ \xi \downarrow & \Downarrow_F & \downarrow \psi \\ X' & \xrightarrow{g'} & Y' \end{array}$$

We take as starting point “double bicategories”, found by taking bicategories internal to **Bicat**, where composition satisfies weaker axioms; in particular there is an associator isomorphism for each composition. This structure turns out to be useful for describing structures build from cospans; in particular, a weak cubical 2-category of cobordisms between cobordisms between manifolds, in any dimension. These are most naturally taken with weak composition, since composition is by gluing of smooth manifolds (by a diffeomorphism).

The author has described the resulting structure elsewhere [Mo], and related it to a related definition given by Dominic Verity. Verity’s double bicategories can be obtained from internal bicategories in **Bicat** in cases where a certain niche-filling condition obtains. We will discuss here how this can be specified as a property of a model, and thus find more restricted notions of cubical n -category as special cases.

First we recap the idea of finite limits theories, and how categories and bicategories, at least (and similarly globular n -categories for any n) are described in

terms of finite limits theories. We note how to obtain double categories, or double bicategories respectively, and describe how this construction generalizes by iterated application of the process of taking models.

That is, given a finite limits theory, say $\mathbf{Th}(\mathbf{Cat})$, the *theory of categories*, one repeatedly applies the functor $[\mathbf{Th}(\mathbf{Cat}), -]$. This notation indicates the category of models of $\mathbf{Th}(\mathbf{Cat})$ in the target category. Thus, one takes categories of models, and then finds models in *these* categories. The structures formed in this way are strict cubical n -categories. Doing the same with the theory of bicategories, $\mathbf{Th}(\mathbf{Bicat})$, gives the particular notion of weak cubical n -category we are discussing, namely, n -tuple bicategories. We will not discuss tricategories, tetracategories, or other forms of n -category here, but note a similar process gives n -tuple tricategories etc.

There are other possible generalizations of this way of generating cubical n -categories. One involves weakening the notion of model: we will not consider this in depth, but will consider the example of pseudocategories to see how this approach motivated the one we are taking here. We discuss will double categories and pseudocategories as strict and weak models of the theory $\mathbf{Th}(\mathbf{Cat})$, and how this motivates double bicategories as models of $\mathbf{Th}(\mathbf{Bicat})$. We will not consider, here, the full extent of weakening possible due to the fact that \mathbf{Bicat} is not just a category, but a tricategory.

We also briefly consider the “niche-filling” conditions which can be used to produce Verity double bicategories, which have fewer different types of morphism. It is possible to extend this to higher n , but the conditions become more complex, and we will regard it as more elegant to leave the extra types of morphism in place, and conclude by indicating an example where this is quite natural.

2. THE THEORY OF BICATEGORIES

To begin with, we recall that, following the approach of Lawvere [WL] to universal algebra (and see also [Bo2]), an algebraic theory can be understood as a category. For example, the theory of groups can be seen as $\mathbf{Th}(\mathbf{Grp})$, the free Cartesian category on a group object. That is, $\mathbf{Th}(\mathbf{Grp})$ is the minimal category with finite products, a terminal object, and an object G equipped with maps $m : G \times G \rightarrow G$ and $e : 1 \rightarrow G$ satisfying the usual group axioms. For example, $(m \otimes 1) \circ m = m \circ (1 \otimes m)$. Then a product-preserving functor $M : \mathbf{Th}(\mathbf{Grp}) \rightarrow \mathcal{C}$, into a Cartesian category \mathcal{C} amounts to the same thing as a “group object in \mathcal{C} ” in the usual sense. That is, the group object is $M(G)$, and the multiplication and unit maps are $M(m)$ and $M(e)$. In the case that $\mathcal{C} = \mathbf{Sets}$, such a functor is just a group.

A theory for a structure \mathcal{S} is described by a category $\mathbf{Th}(\mathcal{S})$ which we think of as a *diagram* containing all the axioms defining \mathcal{S} . A *model* of such a theory in a category \mathbf{C} is a functor into \mathbf{C} .

Some structures cannot be described by *algebraic* theories, however. These categories need only have objects $\{T^0, T^1, T^2, \dots\}$, the powers of a given object, such as the object $G \in \mathbf{Th}(\mathbf{Grp})$. Taking models $F : \mathbf{Th}(\mathbf{Grp}) \rightarrow \mathbf{Set}$, one then gets the usual definition of a group as the set $F(G)$, equipped with set maps such as $F(m) : F(G) \times F(G) \rightarrow F(G)$, satisfying the axioms. However, some structures—categories themselves, for example—are not naturally described in terms of a single set (i.e. a single object in the theory, and its powers). This raises the question of

the *doctrine* of the theory—that is, the specific 2-category of categories in which we take the theory, and the functors which are its models. The doctrine specifies what structures are required to define the theory (as products, or at least monoidal structure, are required to define the multiplication map for the theory of groups). These are the structures which must be preserved by the model functors.

To begin describing n -categories in terms of models, we need to describe the *theory of categories*, and indeed of bicategories. The crucial difference between the (algebraic) theory of groups, and the theory of categories is that the composition operation for morphisms, which plays the role of the multiplication map m in the group example, is only partially defined. This means $\mathbf{Th}(\mathbf{Cat})$ must have some extra structure.

Definition 1. *The doctrine of categories with finite limits is a 2-category \mathbf{FL} whose objects are categories with all finite limits, as morphisms all functors which preserve finite limits, and as 2-morphisms all natural transformations. A finite limits theory is a category \mathcal{T} in \mathbf{FL} , and a model of \mathcal{T} is a functor $\mathcal{T} \rightarrow \mathcal{C}$.*

(Of course, we only usually emphasize that a category is a theory if it is easily presentable in terms of generators and relations, as in the theory of groups.)

We want to describe a whole class of concepts of cubical n -category which can be framed in terms of model theory. These are (some sort of) models of a theory of (globular, or other) n -categories in a suitable category of such n -categories. There are many variants, depending on the choice of theory and the target category for model. We want a “weak” notion, in the sense that composition is associative up to an isomorphism, rather than exactly. The most elementary higher categories with this property are bicategories.

To begin with, we consider the simpler theory of categories, $\mathbf{Th}(\mathbf{Cat})$. The usual definition of a category will amount to a model of $\mathbf{Th}(\mathbf{Cat})$ in \mathbf{Set} .

Definition 2. *The theory $\mathbf{Th}(\mathbf{Cat})$ is a category in \mathbf{FL} , generated by the following data:*

- two objects Obj and Mor
- morphisms of the form:

$$(2) \quad Mor \begin{array}{c} \xrightarrow{s} \\ \xleftarrow{t} \end{array} Obj$$

and

$$(3) \quad Obj \xrightarrow{id} Mor$$

such that $s(id) = t(id) = 1_{Obj}$ as expected.

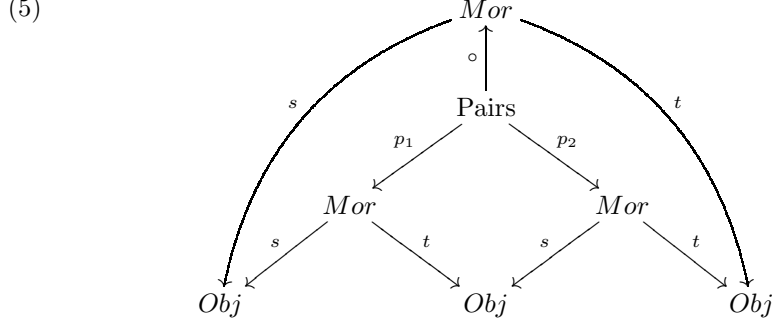
- if Pairs is the pullback in the square:

$$(4) \quad \begin{array}{ccccc} & & \text{Pairs} & & \\ & \swarrow p_1 & & \searrow p_2 & \\ Mor & & & & Mor \\ & \searrow t & & \swarrow s & \\ & & Obj & & \end{array}$$

then there is a (“partially defined”) composition map

$$\circ : \text{Pairs} \rightarrow \text{Mor}$$

satisfying the usual properties for composition, namely:



(and another axiom with the interpretation in **Sets** that for any morphism $f \in \text{Mor}$, we have $\text{id}(s(f))$ and $1(t(f))$ are composable with f , and the composite is f).

Remark 2.0.1. For categories in **Set**, the pullback which gives the object Pairs is the fibered product $\text{Pairs} = \text{Mor} \times_{\text{Obj}} \text{Mor}$, which gives the usual interpretation as a set of composable pairs. This object exists in general since **Th(Cat)** contains all such finite limits. If F preserves finite limits, $F(\text{Pairs})$ will be a pullback again.

In Section 3.1 we briefly recall how a (small) double category is a model of **Th(Cat)** the *theory of categories* in **Cat**, which is the category whose objects are (small) categories and whose morphisms are functors. However, since we are really interested in *weak* cubical n -categories, we will use the finite limits theory describing *bicategories*, and thus encodes this notion of weakness.

The theory of bicategories, **Th(Bicat)**, is more complicated than **Th(Cat)**, but having seen **Th(Cat)** we can abbreviate its description somewhat.

Definition 3. The theory of bicategories **Th(Bicat)** is the category with finite limits generated by the following data:

- *Objects:* Ob, Mor, 2Mor

- *Morphisms:*

Source and target maps:

- $s, t : \text{Mor} \rightarrow \text{Ob}$
- $s, t : 2\text{Mor} \rightarrow \text{Ob}$
- $s, t : 2\text{Mor} \rightarrow \text{Mor}$

Composition maps:

- $\circ : \text{MPairs} \rightarrow \text{Mor}$
- $\circ : \text{HPairs} \rightarrow 2\text{Mor}$
- $\cdot : \text{VPairs} \rightarrow 2\text{Mor}$

where

- $\text{MPairs} = \text{Mor} \times_{\text{Ob}} \text{Mor}$
- $\text{HPairs} = 2\text{Mor} \times_{\text{Mor}} 2\text{Mor}$
- $\text{VPairs} = 2\text{Mor} \times_{\text{Ob}} 2\text{Mor}$

are given by pullbacks as in **Th(Cat)**

– The **associator** map

$$a : \text{Triples} \rightarrow 2\text{Mor}$$

such that a makes the following diagram commute:

$$\begin{array}{ccccc} \text{Pairs} & \xleftarrow{\circ \times 1} & \text{Triples} & \xrightarrow{1 \times \circ} & \text{Pairs} \\ \downarrow \circ & & \downarrow a & & \downarrow \circ \\ \text{Mor} & \xleftarrow{s} & 2\text{Mor} & \xrightarrow{t} & \text{Mor} \end{array}$$

(and additional diagrams with the interpretation that a gives invertible 2-morphisms).

– **unitors**

$$l, r : \text{Ob} \rightarrow \text{Mor}$$

with the obvious conditions on source and target maps.

This data is subject to the usual conditions, including composition rules for 2-morphisms similar to those for morphisms in $\mathbf{Th}(\mathbf{Cat})$, as well as the fact that the compositions for 2-morphisms satisfy the interchange law, associator is subject to the Pentagon identity, and the unitors obey certain unitor laws.

The preceding being terse, we note that the Pentagon identity for a bicategory (i.e. a bicategory in **Sets**) is generally described by saying that the two ways for a composite of associators from $f \circ (g \circ (h \circ k))$ to $((f \circ g) \circ h) \circ k$ are equal. We can express this condition formally, in any category with pullbacks, building from composable quadruples. The pentagon identity may be expressed by a commuting diagram which is given in [Mo], though that paper omits explicit mention of $\mathbf{Th}(\mathbf{Bicat})$. There are similar diagrams for unitor laws. Commutativity of these diagrams is imposed in $\mathbf{Th}(\mathbf{Bicat})$.

We do not propose here to explicitly discuss the theories of tricategories (though see Gordon, Power and Street [GPS], and the appendix in Gurski [NG]), tetracategories (though see Trimble [TT]), and so forth. Explicit descriptions of these theories become quite elaborate very quickly. These and various other higher categorical structures have been discussed extensively elsewhere by Cheng and Lauda [CL], and Leinster [Le, Le2]. We do note, however, that, at least the usual, fairly well understood, definitions of globular n -categories these definitions can be cast as finite limits theories $\mathbf{Th}(\mathbf{Tricat})$ and $\mathbf{Th}(\mathbf{Tetracat})$, and so on. So the process we shall describe here can be applied to all these theories, giving structures which are correspondingly weaker. With bicategories, we have the first case where “weak” is meaningful.

Having described the finite limits theory of bicategories, we consider how to use these theories to define cubical n -categories.

3. MODELS AS CUBICAL n -CATEGORIES

We have described the theories of categories and bicategories. Our idea here is to see how cubical n -categories can be built by taking models of them in the appropriate setting.

3.1. Models of Categories and Bicategories.

Definition 4. A model of a finite limits theory \mathbf{T} in a category \mathbf{C} with finite limits is a finite limit-preserving functor

$$F : \mathbf{T} \rightarrow \mathbf{C}$$

To begin with, we will consider $\mathbf{T} = \mathbf{Th}(\mathbf{Cat})$, and see how to describe the operation of taking “ n -fold categories”, a form of strict n -category.

A (small) category is a model of the theory $\mathbf{Th}(\mathbf{Cat})$ in \mathbf{Set} . That is, it is a functor $F : \mathbf{Th}(\mathbf{Cat}) \rightarrow \mathbf{Set}$, which is specified by choosing a set of objects and a set of morphisms, together with set maps making these into a category. In this setting, the pullback construction means that when the target of a morphism f is the source of g , there is a composite morphism $g \circ f$ from the source of f to the target of g just as expected.

The theory of categories encodes the usual category axioms in terms of commuting diagrams in $\mathbf{Th}(\mathbf{Cat})$. Thus the axioms, in the presentation of a theory, amount to imposing relations between the arrows. For instance, the axiom for associativity can be expressed by the commuting diagram:

$$(6) \quad \begin{array}{ccc} \text{Triples} & \xrightarrow{\circ \times id} & \text{Pairs} \\ \downarrow \circ & & \downarrow id \times \circ \\ \text{Pairs} & \xrightarrow{\circ} & \text{Mor} \end{array}$$

Since $\mathbf{Th}(\mathbf{Cat})$ has all finite limits, there are objects denoting composable k -tuples of morphisms for each k , similar to Pairs, such as

$$(7) \quad \text{Triples} = \text{Mor} \times_{\text{Ob}} \text{Mor} \times_{\text{Ob}} \text{Mor}$$

This is a model of $\mathbf{Th}(\mathbf{Cat})$ in \mathbf{Set} . Our motivating idea is to consider models of $\mathbf{Th}(\mathbf{Cat})$ in the target category $\mathbf{C} = \mathbf{Cat}$. Such a model F gives a category $\mathbf{Ob} = F(\text{Obj})$ of “objects” and a category $\mathbf{Mor} = F(\text{Mor})$ of “morphisms”, with functors s and t , Id , and \circ satisfying the usual category axioms. Note that these axioms give conditions at both the object and morphism level, in addition to those which follow from the fact that they are functors. Functoriality means that there are compatibility conditions between the categorical structures in the two directions. In fact, these amounts to precisely the definition of a *double category*. The “horizontal” category is \mathbf{Ob} and the “vertical” category consists of the objects in \mathbf{Ob} and \mathbf{Mor} together with the object maps from the functors s , t , and so forth. The square 2-cells of the double category are the morphisms of \mathbf{Mor} . It can readily be checked that this gives the usual notion of a double category. (This is discussed in Leinster [Le]).

Moreover, a natural transformation $\nu : F \rightarrow G$ between two models $F, G : \mathbf{Th}(\mathbf{Cat}) \rightarrow \mathbf{Cat}$ is just a double functor in the usual sense. In particular, there are functors $\nu(\text{Ob}) : F(\text{Ob}) \rightarrow G(\text{Ob})$ and $\nu(\text{Mor}) : F(\text{Mor}) \rightarrow G(\text{Mor})$, and similarly for $\text{Pairs} = \text{Mor} \times_{\text{Ob}} \text{Mor}$ and so on. The object and morphism maps of each of these give assignments for the objects, horizontal and vertical morphisms, and squares of the double category. The fact that ν is natural gives compatibility conditions between all these maps and the relevant s , t , Id and \circ which give exactly the fact that these assignments define a double functor (in particular, that there is a functor between the vertical categories).

This describes a *strict* model of $\mathbf{Th}(\mathbf{Cat})$ in \mathbf{Cat} . A *weak* model would satisfy the equations in the category axioms only up to a 2-morphism in \mathbf{Cat} , namely up to natural transformation. As before, there are categories \mathbf{Ob} and \mathbf{Mor} , and functors s , t , Id , and \circ . However, the equations which hold for categories are identities in $\mathbf{Th}(\mathbf{Cat})$, such as associativity, which are mapped to 2-morphisms - that is, natural transformations in \mathbf{Cat} . That is, regarding the category $\mathbf{Th}(\mathbf{Cat})$ as having identity 2-morphisms, a weak model allows equations (identity 2-morphisms) to map to non-identity natural transformations in \mathbf{Cat} . To ensure coherence, this must be done so that any diagrams of such 2-isomorphisms must commute. MacLane's coherence theorem (see e.g. [CWM], for the result in the context of monoidal categories) implies that it is sufficient to have the pentagon identity and unitor identity to imply commutativity of all such diagrams.

So, for instance, composable pairs would be defined by *weak* pullback (in \mathbf{Cat}) rather than strict pullback (as in \mathbf{Set}), so that in the diagram (4), instead of satisfying $t \cdot \pi_1 \cdot i = s \circ \pi_2 \cdot i$, there would only be a natural isomorphism $\alpha : t \cdot \pi_1 \cdot i \rightarrow s \circ \pi_2 \cdot i$. Such a weak model is the most general kind of model available in \mathbf{Cat} , but this does not give a general weak cubical n -category. In particular, it composition is weak in only one direction. This is equivalent to the definition of a *pseudocategory* (see, for instance, [MF], and again [Le]).

In particular, the reason we have weaker composition rules in one direction for a pseudocategory than the other, from this point of view, is that taking the target category $\mathbf{C} = \mathbf{Cat}$ determines that the horizontal structures are (strict) categories, while weakening the axioms from $\mathbf{Th}(\mathbf{Cat})$ implies we have vertical bicategories since, for instance, equations for associativity are mapped to associator isomorphisms which satisfy the pentagon identity.

Pseudocategories have a number of natural examples when one has two different types of morphism between the same objects, and in one case, composition is naturally defined only up to isomorphism. For example, there is a pseudocategory whose objects are sets, and where the horizontal category has functions as morphisms, and the vertical category has *spans* of sets, composed by pullback. A related example has rings as objects, homomorphisms as horizontal morphisms, and bimodules (composed by tensor product) as vertical. Square cells, in these examples, consist of maps of the spans, or bimodules, that are compatible with the horizontal maps.

Unfortunately, pseudocategories are only weak in one direction, and strict in the other. Moreover, for reasons of well-formedness, it is impossible to use squares as the 2-morphisms to weaken composition in both directions. Yet in general, we would like a definition which is weak in both directions—in the conclusion we return to a class of examples where this is the natural choice. We return to this in the conclusion. For now, we look at this definition.

3.2. Internal Bicategories. The fact that a pseudocategory contains “vertical bicategories” suggests a generalization of our approach to double categories. This is to consider (strict!) models of $\mathbf{Th}(\mathbf{Bicat})$ in \mathbf{Bicat} —that is, functors $F : \mathbf{Th}(\mathbf{Bicat}) \rightarrow \mathbf{Bicat}$. It will be relatively straightforward to treat $F(\mathbf{Obj})$ as a horizontal bicategory, and the objects of $F(\mathbf{Obj})$, $F(\mathbf{Mor})$ and $F(2\mathbf{Mor})$ as forming a vertical bicategory, but we note that a diagrammatic representation of, for instance, 2-morphisms in $F(2\mathbf{Mor})$ would require a 4-dimensional diagram element. These structures, termed *double bicategories*, are described in [Mo].

Though we have defined them in terms of the theory of bicategories, as with double categories, these structures can also be described in elementary terms. They have nine types of components, namely the objects, morphisms, and 2-morphisms in each of the bicategories $F(\mathbf{Obj})$, $F(\mathbf{Mor})$ and $F(\mathbf{2Mor})$. There are a number of connecting “face maps” derived from the s and t maps, composition rules, and so on. The most natural way to express these diagrammatically involves cells of dimension up to 4, drawn as products of 0-, 1-, and 2-cells. For example, in both horizontal and vertical directions, there are 3-dimensional morphisms like the “pillow” P here:

$$(8) \quad \begin{array}{ccc} x & \xrightarrow{\quad} & y \\ \downarrow f & \searrow \alpha & \downarrow f' \\ x' & \xrightarrow{\quad} & y' \end{array} \quad \Rightarrow_P \quad \begin{array}{ccc} x & \xrightarrow{\quad} & y \\ \downarrow f & \searrow F_2 & \downarrow g \\ x' & \xrightarrow{\quad} & y' \end{array} \quad \begin{array}{ccc} y & \xrightarrow{\quad} & g' \\ \downarrow g & \searrow \beta & \downarrow g' \end{array}$$

Diagrammatically, P should be drawn as the product of an edge and a (globular) 2-cell. The vertical pillows are the morphisms of $F(\mathbf{2Mor})$, while the horizontal pillows are the 2-morphisms of $F(\mathbf{Mor})$

Since one naturally might hope for a fully weak cubical 2-category to have cells of dimension at most 2, it is also convenient that certain double bicategories, satisfying “niche-filling” conditions, give rise to what Dominic Verity previously called double bicategories, and we call “Verity double bicategories”. These have horizontal and vertical bicategories, as well as squares like a double category, whose composition laws in both directions are weakly associative (up to 2-cells, rather than squares as in pseudocategories). We will return to this in Section 3.3. For now, if we instead take the definition of a weak cubical 2-category by internalization as natural, and follow it, we can see how to extend it to n -categories.

Now, a model of a theory \mathbf{T} as a functor $F : \mathbf{T} \rightarrow \mathcal{C}$, hence the maps between models are natural transformations $\nu : F \rightarrow G$. In particular, ν gives, for each object $t \in \mathbf{T}$, a morphism $\nu(t) : F(t) \rightarrow G(t)$; for every morphism in \mathbf{T} , there is a naturality square. So, in particular, this defines a concept of morphism of models, and thus the category of all models, which we denote $[\mathbf{T} \rightarrow \mathcal{C}]$. In the case of a simple algebraic theory such as $\mathbf{Th}(\mathbf{Grp})$, this defines the notion of group homomorphism in any given setting (say, continuous homomorphism between topological groups, if $\mathcal{C} = \mathbf{Top}$).

Now again we consider the strict case. For $\mathbf{T} = \mathbf{Th}(\mathbf{Cat})$, such a ν defines, in particular, maps $\nu(\mathbf{Ob}) : F(\mathbf{Ob}) \rightarrow G(\mathbf{Ob})$ and $\nu(\mathbf{Mor}) : F(\mathbf{Mor}) \rightarrow G(\mathbf{Mor})$, which commute with the maps of \mathbf{T} (in the sense of commuting naturality squares). As we saw, such a ν defines the notion of a functor between categories, internal to \mathcal{C} . In particular, if $\mathcal{C} = \mathbf{Cat}$, this defines the usual notion of a functor between double categories, in the language of models in \mathbf{Cat} .

So consider the functor category $\mathcal{C} = [\mathbf{Th}(\mathbf{Cat}), \mathbf{Cat}]$, whose objects are double categories (models of $\mathbf{Th}(\mathbf{Cat})$ in \mathbf{Cat}), and whose morphisms are double functors (natural transformations ν). The functor category inherits the property of having all finite limits from the target \mathbf{Cat} , since the limit of a diagram of functors, at each object $X \in \mathbf{Th}(\mathbf{Cat})$, gives the limit of the diagram applied to X . So we can take this new \mathcal{C} as our new target category for models of our finite limits theory. Then models $F : \mathbf{Th}(\mathbf{Cat}) \rightarrow \mathcal{C}$ are triple categories: that is, cubical 3-categories.

Natural transformations between such models are triple-functors in a natural sense, and this gives a new category. Iterating this process gives the usual notion of strict cubical n -category as an n -fold category.

The analogous process for $\mathbf{T} = \mathbf{Th}(\mathbf{Bicat})$ and $\mathcal{C} = \mathbf{Bicat}$, the (1-)category whose objects are bicategories and whose morphisms are homomorphisms (2-functors) between them. Then double bicategories may be seen as functors $F : \mathbf{T} \rightarrow \mathcal{C}$, and natural transformations between these models give a notion of double bifunctor. Then there is a functor category $[\mathbf{T}, \mathcal{C}]$. Taking this to be our new \mathcal{C} , and iterating the process, we get a notion of weak cubical n -category for all n as an n -tuple bicategory. In particular, we can inductively define:

Definition 5. *A weak cubical 0-category is just a set, and a functor between these is a set function, so we say $0 - \mathbf{tupleBicat} = \mathbf{Set}$. Given a category $(n - 1) - \mathbf{tupleBicat}$, define $n - \mathbf{tupleBicat} = [\mathbf{Th}(\mathbf{Bicat}), (n - 1) - \mathbf{tupleBicat}]$.*

To describe these more completely, note that we can inductively describe all the types of data which make up an n -tuple bicategory, given by the model F . In particular, there will be a natural interpretation where we have 3^n types of morphism (including objects as 0-morphisms). This is because there is one type of data (elements) when $n = 0$; and for $n > 0$, we have $(n - 1)$ -tuple bicategories $F(\mathbf{Ob})$, $F(\mathbf{Mor})$ and $F(2\mathbf{Mor})$, each with 3^{n-1} types of data. Diagrammatically, these can be represented as n -fold products of dot, edge, and globular 2-cell (indicating which type we select at each step of the induction). Thus, the morphisms naturally have dimension up to $2n$, although they are composable in only n directions. Each dimension corresponds to one stage of the inductive construction.

The composition rules follow from the fact that the composition in any direction is given by the bicategory axioms in the corresponding stage of the construction. For instance, composition in each direction for cubes (products of edges) is weakly associative in each direction, where the associator is a morphism given as a product of $(n - 1)$ edges (in all other directions), and a 2-cell (in the direction of composition). On the other hand, composition of 2-cells in a bicategory (i.e. within the category $\mathbf{hom}(x, y)$) is strict. So composition of those morphisms given as products of a 2-cell with other data will be strict in the corresponding direction. However, it makes sense to say that an n -tuple bicategory is a “weak cubical n -category”, since at least the cubes have weakly associative composition in all n directions.

We have remarked that the term “double bicategory” was used by Verity to describe a somewhat different structure, in which all morphisms are naturally represented as 2-dimensional. Since it seems reasonable to expect that an n -category should have morphisms of dimension at most n , we briefly consider how the two are related.

3.3. Niche-filling Conditions. It is an impetus for much research that there are various relations between different definitions of n -category. In particular, relations between (strict) cubical and globular n -categories have been described, by Brown and Higgins [BH], among others. So for example, double categories are related to 2-categories: if horizontal and vertical morphisms can be composed, then squares can be considered to be 2-cells between the composites of the edges. For higher n , and weaker notions of n -category, more complex relations become possible. In [Mo], we discussed “niche-filling conditions” which reduced a double bicategory in the sense of a model of $\mathbf{Th}(\mathbf{Bicat})$ in \mathbf{Bicat} , to a Verity double bicategory. Such

a structure has horizontal and vertical bicategories, as well as squares, together with various composition rules, and also actions of 2-morphisms on squares. Here we briefly give an account of these niche-filling conditions in terms of models of a finite-limits theory.

A niche-filling condition states that, given any suitable combination of cells (that is, objects, morphisms, 2-morphisms, and so on) of a particular shape, there is some other cell which “fills the niche” by completing the diagram in a specified way. A simple example of a niche-filling condition is fulfilled by the composition operation for morphisms in a category (i.e. model of $\mathbf{Th}(\mathbf{Cat})$ in \mathbf{Set}). Here, the niche is given by a choice of object x and pair of “composable” morphisms f and g with $s(g) = x$ and $t(f) = x$. Then the filler for this niche is the composite $g \circ f : s(f) \rightarrow t(g)$.

Such conditions play a major role in definitions of simplicial n -categories (generalized to Joyal’s “quasicategories” [Jo], also called ∞ -categories by Lurie [Lu]) in which the various axioms for a category amount to “horn-filling conditions”. For example, composition is replaced by a 2-simplex (a 2-morphism in a simplicial n - or ∞ -category), or rather the condition that, for morphisms f and g with $t(f) = s(g)$, there is a triangle C and edge $g \circ f$ filling the diagram:

$$(9) \quad \begin{array}{ccc} & y & \\ f \nearrow & \Downarrow_F & \searrow g \\ x & \dashrightarrow & z \\ & g \circ f & \end{array}$$

Other properties for such ∞ -categories are also expressed as horn-filling conditions

In the cubical case, the more complex shapes of the niches and the greater number of distinct operations make the situation slightly trickier, but they are also less crucial to our chosen definition. They do, however, give modifications to it. An example of a useful niche-filling condition in a double category \mathcal{D} would be the following. Suppose that, given a horizontal arrow f and vertical arrow g in \mathcal{D} , where the source of g is the target of f , there is a unique invertible square F and vertical arrow h making the following commute:

$$(10) \quad \begin{array}{ccc} X & \xrightarrow{f} & Y \\ h \downarrow & \Downarrow_F & \downarrow g \\ Z & \xrightarrow{\text{id}} & Z \end{array}$$

Then one can define h to be the composite $g \circ f$, and get a category generated by the morphisms of both **Hor** and **Ver**, where the F in the above is regarded as the identity, and all other squares are discarded. In fact, to do this, we do not necessarily need that there is a unique such “niche-filler”, only that there is a specified way to find one, which we can use to define composition (and that these choices are coherent).

There are analogous conditions for double bicategories: horizontal and vertical “action conditions”, and a compatibility condition, which turn the intrinsically four-dimensional structure into a 2-dimensional structure satisfying Verity’s definition of “double bicategory”. As with double categories in the example above, a bicategory can be obtained from a double bicategory by allowing composition of horizontal and vertical morphisms. Here, we are less interested with this, than with

imposing conditions which give a new notion of n -category from an old one, by discarding certain higher morphisms (in this case, the 3- and 4-dimensional ones), in a consistent way. In particular, the condition of interest specifies *actions of 2-cells on squares*. Given a 2-cell α and a square F_1 which share an edge (i.e. are composable), the condition allows one to complete the left half of the “pillow” diagram (8) with three data. These are a (unique, invertible) 3-dimensional cell F , the opposite square F_2 , taken as the “composite” $\alpha \circ F_1$, and with the 2-cell $\beta = \text{id}$.

In general, a niche-filling condition demands, given a certain collection of cells (i.e. data from the final model in the chain), that there are other cells which compose with them, satisfying some commutation conditions. A strong version of a niche-filling condition requires that such niche-filling data are unique. A weaker version merely requires that some filler exist. The sort of condition we want is one which specifies a filler given a niche: that is, we require that our weak n -category be equipped with maps specifying the fillers of any niche. This includes the case where there is a unique filler. Given such fillers, we can define actions of one type of morphism on another by taking the fillers between them to be “thin”. That is, we consider them to be the identity, and discard all other morphisms of the chosen shape.

Given such a niche-filling condition, we can consider the category of all models which are equipped with such a map. Provided these categories have finite limits (i.e. that the niche-filling condition is preserved by taking finite limits), we can use them in our process of taking iterated models. Starting with $[\mathbf{Th}(\mathbf{Bicat}), \mathbf{Bicat}]$, the category of double bicategories in our sense, we can find a category consisting of models in this category *equipped with* a map giving niche-fillers of the kind used in the three action conditions. Each of these determines a double bicategory in the sense of Verity. Functors which preserve the niche-filling maps make these into a category **VDB**. In fact, **VDB** contains finite limits, so we can take $[\mathbf{Th}(\mathbf{Bicat}), \mathbf{VDB}]$.

Now, a model $F \in [\mathbf{Th}(\mathbf{Bicat}), \mathbf{VDB}]$ determines Verity double bicategories $F(\text{Ob})$, $F(\text{Mor})$, and $F(2\text{Mor})$ and the connecting double functors such as $F(\circ)$, and so forth. This can be interpreted as a (weak) cubical 3-category. Moreover, it is weaker in the new direction, since there are higher-dimensional cells representing, for example, squares in $F(2\text{Mor})$, which would be four-dimensional. If we want our notion of weak n -category to have morphisms represented by cells of dimension at most n , we again need a niche-filling conditions here which would specify which of these cells to use when pasting.

That is, we would need to specify cells of various dimensions which define the niche to be filled. One approach, then, is at each step to take the full subcategory of models which satisfy these conditions. However, the number of conditions grows at each step, since there are more directions in which composites need to be defined. A different approach would be to incorporate the niche-filling conditions into our theory. However, this means inverting the process used so far, in which our target category \mathcal{C} for the model F changes, but the theory $\mathbf{Th}(\mathbf{Bicat})$ stays the same. To address niche-filling conditions at the level of the theory, we would need to obtain $\mathbf{Th}(\mathbf{DblBicat})$, a theory of double bicategories (namely, of functors from $\mathbf{Th}(\mathbf{Bicat})$ into \mathbf{Bicat}), and add specified maps which give the niche-fillers, satisfying the implied conditions.

Neither of these approaches to reducing the dimension of the cells of our weak n -categories is particularly elegant, so here we will adopt the view that the fully general definition is in some sense simpler.

4. CONCLUSION

The presentation we have given for weak cubical n -categories suggests the simple definition that they are n -fold bicategories: models given by applying the functor $[\mathbf{Th}(\mathbf{Bicat}), -]^n$ to \mathbf{Set} . Thus, $[\mathbf{Th}(\mathbf{Bicat}), \mathbf{Set}] = \mathbf{Bicat}$ by definition. Then $[\mathbf{Th}(\mathbf{Bicat}), \mathbf{Bicat}]$ can be denoted $\mathbf{DblBicat}$, $[\mathbf{Th}(\mathbf{Bicat}), \mathbf{DblBicat}]$ can be denoted $\mathbf{TrplBicat}$ and so forth. But note that at each step of the iteration, we might make a difference choice of which theory to model in the category produced at the previous step, and of how strict the model should be. In particular, we have not considered the question of what a weak model of $\mathbf{Th}(\mathbf{Bicat})$ in \mathbf{Bicat} would be, but treating \mathbf{Bicat} as a mere category ignores its full structure. \mathbf{Bicat} is most naturally a tricategory in which the morphisms are 2-functors, 2-morphisms are natural transformations, and 3-morphisms are modifications, allowing weak models. So in fact, this schema can generate many different definitions of cubical n -category of many different degrees of strength and weakness. We are restricting attention to strict models, since the structures these produce are already weak enough for some relevant applications.

It is not unusual for different applications to suggest different definitions of n -category, which is one reason for their abundance (see [CL]). In particular, n -fold bicategories are quite natural for extending the classes of examples discussed in [Mo] based on (co)spans (also developed extensively by Grandis [Gr1, Gr3]). This is a generalization of the bicategory of cospans in a category \mathcal{C} with pushouts. Given $X, Y \in \mathcal{C}$, the morphisms in $\mathbf{Cosp}(\mathcal{C})$ from X to Y are diagrams $X \rightarrow S \leftarrow Y$, and 2-morphisms between cospans are “cospan maps” given by morphisms $f : S \rightarrow S'$ in \mathcal{C} which commute with the maps from X and Y . Spans compose by pushout along two common inclusions. In the cubical case, one treats n -fold products of such diagrams. This naturally fits the framework of the n -fold bicategories described here.

A special case is the topological example discussed in [Mo], which was the main motivation there. Here, the cospans of interest are *cobordisms* between manifolds. That is, S is a manifold with boundary, and the maps from X and Y are inclusions of boundary components (in the smooth case, the extra structure of a collar is needed, which is discussed in a general setting by Grandis). Among other things, cobordisms give a way to study manifolds (and in particular find invariants for them) by factoring them into pieces, dealing with each piece, and then composing the pieces. Typically, the category \mathbf{nCob} is described as having $(n-1)$ -dimensional manifolds as objects, and *diffeomorphism classes* of cobordisms as morphisms. A diffeomorphism of a cobordism, fixing the boundary (and its collar, if it has one), is just a cospan map in $\mathbf{Cosp}(\mathbf{Man})$. Now, in particular, taking cobordisms, not equivalence classes, as morphisms means that composition (by gluing cobordisms at boundary components) is only weakly associative.

This is why a weak structure was desired. The need for a cubical n -category comes from a generalization motivated by applications to (topological) quantum field theories. This is a generalization to cobordisms *between* cobordisms—in particular, cobordism between manifolds with boundary (suitable for field theories in

backgrounds with boundary conditions). In particular, as a double bicategory, the objects are $(n - 2)$ -dimensional manifolds, the (horizontal and vertical) morphisms are $(n - 1)$ -dimensional cobordisms, and the squares are n -dimensional cobordisms with corners. However, there are other types of morphism here. Horizontal and vertical 2-cells are diffeomorphisms of horizontal and vertical cobordisms. There are also diffeomorphisms of the n -dimensional body. Those which fix (pointwise) the horizontal source are our “vertical pillows”, and vice versa. Those which fix only objects are the top-dimensional cells in the double bicategory.

This can of course be extended to a k -fold bicategory of cobordisms with corners having codimension k between the objects, $(n - k)$ -dimensional manifolds, and the top-level morphisms, the n -dimensional cobordisms with corners, up to and including the case when $n = k$. In each case, we again have cobordisms as the cubical-shaped morphisms, and diffeomorphisms (fixing various components of boundaries) as the remaining morphisms. Each type of morphism appearing in our n -fold bicategories has a natural interpretation.

It is possible, even convenient, to discard some of the complexity of the double bicategory by taking n -cobordisms only up to diffeomorphism (this guarantees the niche-filling conditions discussed earlier, and yields a Verity double bicategory, which itself can be further reduced to a bicategory). But the most natural way to describe the full structure of cobordism is to leave in these morphisms. This means the most natural organizing structure is the one perhaps most simply understood through the process of internalization we have described here—that is, in terms of models of finite limits theories.

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